Proving strict monotonicity of a function in the given set - answers

Read the definition on strict monotonicity of a function.

Definition

A number function f is **strictly monotonic** in a set S if it is either strictly **monotonically increasing** or **strictly monotonically decreasing** throughout S.

- f is **strictly monotonically increasing** in S if for all $x_1, x_2 \in S$ with $x_1 < x_2$, holds that in $f(x_1) < f(x_2)$
- f is strictly **monotonically decreasing** in S if for all $x_1, x_2 \in S$ with $x_1 < x_2$, holds that in $f(x_1) > f(x_2)$

Task 1. (3p) Prove that the function given with the formula $f(x) = \frac{3x-16}{x-5}$.
is strictly monotonic in the interval $(5, \infty)$. What kind of monotonicity is it, increasing or decreasing?

Proof

Let assume that $x_1, x_2 \in (5, \infty)$ and assume that $x_1 < x_2$.

Let' s calculate the difference of values:

$$f(x_1) - f(x_2) = \frac{3x_1-16}{x_1-5} - \frac{3x_2-16}{x_2-5} = \frac{(3x_1-16)(x_2-5)-(3x_2-16)(x_1-5)}{(x_1-5)(x_2-5)} =$$

$$f(x_1) - f(x_2) = \frac{x_1 - x_2}{(x_1-5)(x_2-5)}$$

The assumption $x_1, x_2 \in (5, \infty)$
results in the following inequalities:

$$x_1 > 5 \text{ and } x_2 > 5$$

$$x_1 - 5 > 0 \text{ and } x_2 - 5 > 0$$

$$(x_1 - 5)(x_2 - 5) > 0$$

The assumption $x_1 < x_2$ results in the following inequality:

$$x_1 - x_2 < 0.$$

Thus

$$f(x_1) - f(x_2) = \frac{x_1 - x_2}{(x_1-5)(x_2-5)} < 0$$

$$f(x_1) - f(x_2) < 0$$

$$f(x_1) < f(x_2).$$

This means that f is strictly monotonically increasing in the interval $(5, \infty)$.

Task 2. Prove that the function given with the formula $f(x) = \frac{x+10}{x+7}$.
is strictly monotonic in the interval $(-\infty, -7)$.
What kind of monotonicity is it, increasing or decreasing?

Proof

Let $x_1, x_2 \in (3, \infty)$ and $x_1 < x_2$.

Let' s calculate the difference of values:

$$f(x_1) - f(x_2) = \frac{x_1+10}{x_1+7} - \frac{x_2+10}{x_2+7} = \frac{(x_1+10)(x_2+7)-(x_2+10)(x_1+7)}{(x_1+7)(x_2+7)} =$$

$$f(x_1) - f(x_2) = \frac{-3(x_1 - x_2)}{(x_1+7)(x_2+7)}$$

The assumption $x_1, \; x_2 \in (-\infty, -7)$

results in the following inequalities:

$x_1 < -7$ and $x_2 < -7$

$x_1 + 7 < 0$ and $x_2 + 7 < 0$,

$(x_1 + 7)(x_2 + 7) > 0$.

The assumption $x_1 < x_2$ results in the following inequalities:

$x_1 - x_2 < 0$.

$-3(x_1 - x_2) > 0$

Thus

$$f(x_1) - f(x_2) = \frac{-3(x_1 - x_2)}{(x_1+7)(x_2+7)} > 0$$

$$f(x_1) - f(x_2) > 0$$

$$f(x_1) > f(x_2).$$

This means that f is strictly monotonically decreasing in the interval $(-\infty, -7)$.